10.4 Areas and Lengths in Polar Coordinates

If we have a polar curve, **r = f(Θ)** and $\boldsymbol{a \leq \theta \leq b}$, then the area **A** of the polar region is:

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta$$

Since **r = f(Θ),** we get

$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

Example: Find the area enclosed by $\boldsymbol{r = -5\,sin(\theta)}$ $for\ \mathbf{0} < \boldsymbol{\theta} < \mathbf{2}\boldsymbol{\pi}$.

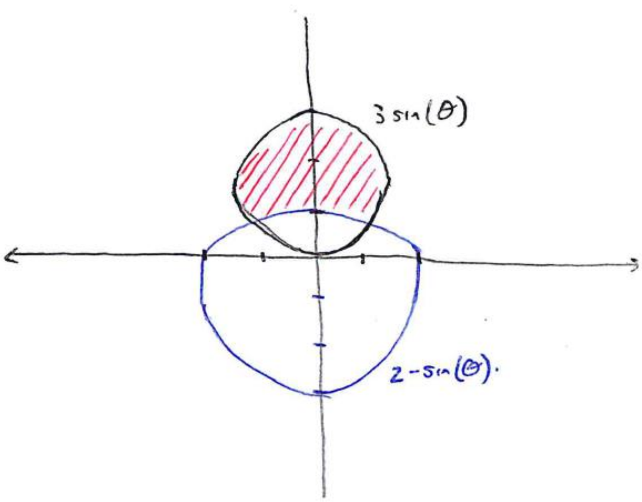
$$A = \int_0^{2\pi} \frac{1}{2}[-5\sin(\theta)]^2 d\theta = \frac{1}{2}\int_0^{2\pi} 25 sin^2(\theta) d\theta = \frac{25}{2}\int_0^{2\pi} sin^2(\theta) d\theta \quad (using\ \cos(2\theta) = 1 - 2sin^2(\theta))$$

$$= \frac{25}{2}\int_0^{2\pi}\left(\frac{1}{2} - \frac{\cos(2\theta)}{2}\right) d\theta = \frac{25}{2}\left[\int_0^{2\pi}\frac{1}{2}d\theta - \frac{1}{2}\int_0^{2\pi}\cos(2\theta)d\theta\right] = \frac{25}{2}\left[\frac{1}{2}\theta - \frac{1}{2}\frac{\sin(2\theta)}{2}\right]_0^{2\pi}$$

$$= \frac{25}{2}\left[\frac{1}{2}\theta - \frac{\sin(2\theta)}{4}\right]_0^{2\pi} = \frac{25}{2}\left[\left(\frac{2\pi}{2} - 0\right) - 0\right] = \mathbf{\frac{25\pi}{2}}$$

Example: Find the area of the region that lies inside the first curve and outside the second curve.

r = 3sin(Θ) and **r = 2 – sin(Θ)** These are two overlapping circles. (See the graph below)



We are interested in the region in red. We need to find where **3sin(Θ)** and **2 – sin(Θ)** intersect.

$$3\sin(\theta) = 2 - \sin(\theta) \quad solve\ for\ \theta$$

$$4\sin(\theta) = 2$$

$$\sin(\theta) = \frac{1}{2}$$

$$\boldsymbol{\theta = \frac{\pi}{6}\ and\ \frac{5\pi}{6}}$$

$$A = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}[3\sin(\theta) - (2 - \sin(\theta))]^2 d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [4\,sin(\theta) - 2]^2 d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [16sin^2(\theta) - 8\sin(\theta) + 4]d\theta$$

$$= \frac{1}{2}\left[16\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} sin^2(\theta)d\theta - 8\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin(\theta)d\theta + \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 4d\theta\right]$$

$$= \frac{1}{2}\left[16\left(\frac{1}{2}\theta - \frac{1}{2}\cdot\frac{\sin(2\theta)}{2}\right) + 8\cos(\theta) + 4\theta\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = \frac{1}{2}[8\theta - 4\sin(2\theta) + 8\cos(\theta) + 4\theta]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$= \frac{1}{2}[12\theta - 4\sin(2\theta) + 8\cos(\theta)]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$= \frac{1}{2}\left[\left(12\left(\frac{5\pi}{6}\right) - 4\sin\left(\frac{5\pi}{3}\right) + 8cos\left(\frac{5\pi}{6}\right)\right) - \left(12\left(\frac{\pi}{6}\right) - 4sin\left(\frac{\pi}{3}\right) + 8cos\left(\frac{\pi}{6}\right)\right)\right]$$

$$= \frac{1}{2}\left[\left(10\pi - 4\left(-\frac{\sqrt{3}}{2}\right) - 8\left(\frac{\sqrt{3}}{2}\right)\right) - \left(2\pi - \frac{4\sqrt{3}}{2} + 8\left(\frac{\sqrt{3}}{2}\right)\right)\right]$$

$$= \frac{1}{2}\left[\left(10\pi + 2\sqrt{3} - 4\sqrt{3}\right) - \left(2\pi - 2\sqrt{3} + 4\sqrt{3}\right)\right] = \frac{1}{2}\left[8\pi - 4\sqrt{3}\right] = \boldsymbol{4\pi - 2\sqrt{3}}$$

The Length of a Polar Curve:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example: Find the exact length of the polar curve $\boldsymbol{r = 2cos(\theta)}$ $\boldsymbol{0 \leq \theta \leq \pi}$

$$\frac{dr}{d\theta} = -2\sin(\theta), \qquad a = 0 \; and \; b = \pi$$

$$L = \int_0^{\pi} \sqrt{(2\cos(\theta))^2 + (-2\sin(\theta))^2} d\theta = \int_0^{\pi} \sqrt{4(cos^2(\theta) + sin^2(\theta))} d\theta = \int_0^{\pi} \sqrt{4} d\theta$$

$$= (2\theta)_0^{\pi} = \boldsymbol{2\pi}$$

As a check, note that the curve is a circle of radius 1, so its circumference is $2\pi \cdot 1 = 2\pi$.